Kinetic Models for Waves in Random Media & Related Inverse Problems

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Motivations

- Imaging in Highly Heterogeneous Media
  - Statistical Properties of Random Media
  - Imaging of buried Inclusions
Probing heterogeneous media

Turbulent Atmosphere

Source

Detector
Detecting Buried Inclusions
Imaging in Known Media

- When heterogeneous medium is known: Use **Time Reversal**: 
  - Time reversed waves back-propagate to their original location. 
  - Inclusion may be seen as secondary source.

- This is the rationale for standard methods such as Kirchhoff migration leading to resolution of order $\lambda/2$. 
Imaging in Unknown Media

When the (random) medium is *not known*:

- Model random medium by a *homogeneous medium* with *small* random fluctuations.

- **Model** wave propagation *macroscopically*:
  
  What we are interested in today.
Homogeneous medium: Kirchhoff migration

Weakly Scattering

Strongly Scattering
High frequency waves in Random Media

- Macroscopic model: need an asymptotic regime. Here **high frequency** waves with **highly heterogeneous** media.

- High frequency waves: **Liouville** equation for the wave **energy density** $a(t, x, k)$:

$$\frac{\partial a}{\partial t} + \nabla_k \omega \cdot \nabla_x a - \nabla_x \omega \cdot \nabla_k a = 0$$

$$\omega(x, k) = c(x)|k|$$
Radiative Transfer Equation

Regime: fluctuations too large to be ignored. Perturbations account for SCATTERING.

\[
\frac{\partial a}{\partial t} + \nabla_k \omega \cdot \nabla_x a - \nabla_x \omega \cdot \nabla_k a = \frac{\pi \omega^2(x, k)}{2(2\pi)^d} \times \int_{\mathbb{R}^d} \hat{R}(x, p - k) \left( a(p) - a(k) \right) \delta \left( \omega(x, p) - \omega(x, k) \right) dp
\]

\( \hat{R}(x, k) \): Power Spectrum of velocity fluctuations
Regimes of Wave propagation

- **Weak Coupling** regime: \( \delta c_\varepsilon^2(x) = \sqrt{\varepsilon} \delta c^2 \left( \frac{x}{\varepsilon} \right) \)
  \[
  \hat{R}(k) \delta(k + p) = c_d \mathbb{E}\{\delta c^2(k) \delta c^2(p)\}
  \]

- **Low Density** regime: \( \hat{R}_0 = c_d \mathbb{E}\{\tau^2\} n_0 \)
  \[
  \delta c_\varepsilon^2(x) = \varepsilon \frac{1 - (\gamma + \beta)}{2} \sum_j \tau_j \delta c^2 \left( \frac{x}{\varepsilon} - \frac{x_\varepsilon^j}{\varepsilon \beta} \right)
  \]
  \( x_\varepsilon^j \) Poisson P.P. with density \( \varepsilon^{(\gamma-1)d} n_0 \)

- **For larger fluctuations**, waves may localize.
Inverse Problem

- Imaging the random media and/or buried inclusions becomes an inverse transport problem:

\[
\frac{\partial a}{\partial t} + \nabla_k \omega \cdot \nabla_x a - \nabla_x \omega \cdot \nabla_k a = \frac{\pi \omega^2(x, k)}{2(2\pi)^d} \times \int_{\mathbb{R}^d} \hat{R}(x, p - k) \left( a(p) - a(k) \right) \delta \left( \omega(x, p) - \omega(x, k) \right)
\]

\[
\omega(x, k) = c(x) |k|
\]

- How stable are available measurements?
Statistical Stability
Result: Under appropriate assumptions, the energy density converges, as the wavelength goes to 0, weakly and in probability, to its deterministic limit.

Weakly means we have to average energy over a sufficiently large region compared to the wavelength.

Some results on speed of convergence.

Result shows that the RTE indeed provides a model suitable for inversion: Measurements are fairly independent of the unknown realization of the random medium.
Inverse Waves v. Inverse Transport

Kirchhoff (inverse wave) reconstruction versus Inverse Transport reconstruction
Inverse Waves v. Inverse Transport

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Inverse Waves v. Inverse Transport

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Kirchhoff (inverse wave) reconstruction versus Inverse Transport reconstruction
Random medium and buried inclusions are modeled as constitutive parameters in a transport equation, which models the (macroscopic) wave energy density.

In the high frequency limit, measurements over sufficiently large detectors are (approximately) statistically stable.
Inverse Transport and ill-posedness

- With **spatially resolved** measurements, the inverse transport problem is **severely ill-posed**: Noise is highly amplified which results in **poor resolution**.

- As we saw, measurements are **statistically stable** if averaged over a sufficiently large domain.

- Important to find imaging scenarios that are as much immune to **statistical noise** as possible (High SNR).
Energies and Correlations

- RTE models more general \textit{field-field correlations} (energies when the fields are the same).

\[ C(t, x) = \mathbb{E}\{u_1(t, x)u_2^*(t, x)\}. \]

- Applications: monitor \textit{turbulent region} as a function of time; image time-varying buried inclusions (field 1 with inclusion; 2 without).
Generalized RTE for Correlations

**Correlation Function**

\[ C(t, x) = \int_{\mathbb{R}^d} a(t, x, k) \, dk \]

\[
\frac{\partial a}{\partial t} + c_0 \hat{k} \cdot \nabla a + (\Sigma(k) + i \Pi(k)) a
\]

\[
= \frac{\pi \omega^2_+(k)}{2(2\pi)^d} \int_{\mathbb{R}^d} \hat{R}_{12}^{12}(k - q) a(q) \delta\left(\omega_+(q) - \omega_+(k)\right) \, dq
\]

\[
\Sigma(k) = \frac{\pi \omega^2_+(k)}{2(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{R}_{11}^{11} + \hat{R}_{22}^{22}}{2} (k - q) \delta\left(\omega_+(q) - \omega_+(k)\right) \, dq
\]

\[
i \Pi(k) = \frac{i \pi \sum_{j=\pm} \phantom{\int_{\mathbb{R}^d}} \not{p.v.} \int_{\mathbb{R}^d} \left( \hat{R}_{11}^{11} - \hat{R}_{22}^{22} \right)(k - q) \frac{\omega_j(k) \omega_+(q)}{\omega_j(q) - \omega_+(k)} \, dq
\]
Imaging Scenarios

- **Scenario 1**: Image from *Direct Energy* Measurements (with inclusion)

- **Scenario 2**: Image from *Energy* Measurements *With and Without* Inclusion

- **Scenario 3**: Image from *Wave Field* Measurements *With and Without* Inclusion
Direct versus Differential Measurements

- Scenario 1 suffers from large statistical instability caused by our lack of knowledge of the random medium.

- Scenarios 2&3 suffer from statistical instability proportional to changes in the differential measurements.
Direct Measurements

- Statistical Instability from Object
- Statistical Instability from Medium
Differential Measurements
Correlations vanish at the inclusion’s boundary

Incompatible

Dispersion Relations
Energies versus Correlations

- Comparison of Scenarios 2&3 in Highly Scattering regime:

In highly scattering media (in the diffusive regime), the perturbation in the energy caused by a void inclusion is given by

\[ \delta \mathcal{E}(t, x) = d\pi D_0 R^d \int_0^t \nabla_x u_0(t - s, x_b) \cdot \nabla_{x_b} G(s, x, x_b) ds. \]

Here \( d \) is dimension and \( G(s, x, x_b) \) the background Green's function.

The perturbation of the two-field correlation is given by

\[ \delta C(t, x) = -4\pi R \int_0^t u_0(t - s, x_b) \frac{G(s, x, x_b)}{s} ds + o(R), \quad d = 3 \]

\[ \delta C(t, x) = \frac{2\pi}{\ln R} \int_0^t u_0(t - s, x_b) \frac{G(s, x, x_b)}{s} ds + o\left(\frac{1}{\ln R}\right), \quad d = 2. \]

- In moderately scattering regime, both are of order \( R^{d-1} \).
Numerical Simulations

- Waves solved by Finite Differences
- Transport solved by Monte Carlo
Typical Wave field
Effect of Void inclusions

- **Transport theory** accurately predicts the influence of an inclusion on the energy measurement.
Noise v. detector size

Wavelength $\lambda = 1$, mean free path $\approx 40$, isotropic cross section.

Display of $S(t) = \frac{\sigma\{\mathcal{E}^\varepsilon\}(t)}{\mathbb{E}\{\mathcal{E}^\varepsilon\}(t)}$ for 20 realizations.

Relative standard deviation for several detectors

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Correlation fluctuations (blue) versus energy fluctuations (red) in weakly (left) and strongly (right) scattering media.
Reconstructions from Energies

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Reconstructions from Correlations

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Inverse monochromatic transport

- Monochromatic waves
- **Foldy Lax** to model **point scatterers** and solve for wave fields
- Forward and inverse transport problems solved by **Monte Carlo** method
- Random medium parameterized by **mean free path**:

$$l_{2D}^*(k) \approx \frac{1}{\tau^2 k^3}$$
Weak Scattering reconstructions

- Kirchhoff (middle) versus Transport (right) reconstructions
Strong Scattering reconstructions

- **Kirchhoff** (middle) versus **Transport** (right) reconstructions
Reconstruction from Direct Measurements

- Inclusion of radius $R=30$
Reconstruction from Differential Measurements

- Inclusion of radius $R=10$
Hidden Inclusions (by known blocker)

- Reconstruction of inclusions in the absence of line of sight (coherent) measurements.
Duke U. experimental Setup

Antenna

Target
Reconstructions from Experimental Data

- Reconstructions based on differential data (Scenario 2).
- 10 GHz data. Medium is 2.5 mean free paths thick.
Reconstruction of voids

- Reconstructions based on differential data (Scenario 2).
- 10 GHz data. Medium is 2.5 mean free paths thick.
Conclusions

- **Transport equations** offer an accurate *macroscopic* description of wave propagation in unknown heterogeneous media.

- **Energy density** and the **field-field correlations** are asymptotically *statistically stable*.

- **Inverse transport** a good model to *reconstruct* statistical properties of random media and *image* buried inclusions.
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References: